

Critical entanglement spectrum of one-dimensional symmetry protected topological phases

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Under an appropriate symmetric extensive bipartition in a one-dimensional symmetry protected topological (SPT) phase, a bulk critical entanglement spectrum can be obtained, resembling the excitation spectrum of the critical point separating the SPT phase from the trivial (vacuum) state. Such a critical point is beyond the standard Landau-Ginzburg-Wilson paradigm for symmetry breaking phase transitions. For the $S = 1$ SPT (Haldane) phase with the Affleck-Kennedy-Lieb-Tasaki exact wave function, the resulting critical entanglement spectrum has a residual entropy per lattice site $s_r = 0.67602$, showing a delocalized version of the edge excitations in the SPT phase. From the wave function corresponding to the lowest entanglement energy level, the central charge of the critical point can be extracted $c \approx 1.01 \pm 0.01$. The critical theory can be identified as the same effective field theory as the spin-1/2 antiferromagnetic Heisenberg chain or the spin-1/2 Haldane-Shastry model with inverse square long-range interaction.

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Introduction.— Topological properties of low-dimensional quantum many-body systems have been attracting considerable interest in quantum information sciences, condensed matter physics and quantum field theories. It is understood that important information of a topological phase is encoded in the von Neumann entanglement entropy of its ground state[1–3]. In a seminal paper, Li and Haldane[4] introduced the entanglement spectrum (ES) from the eigenvalues of the reduced density matrix upon tracing out a subsystem, and the low-lying part bears a remarkable similarity to the physical edge spectrum of the topological state[5–8]. Recently, Hsieh and Fu[9] have suggested that a symmetric extensive bipartition of the ground state wave function for a topological phase leads to a bulk critical ES that resembles the excitation spectrum of the critical point separating the topological phase from the trivial gapped phase. They used an example of noninteracting fermion Chern insulator to illustrate the emergent critical ES to describe the phase transition in the integer quantum Hall effect.

In this Letter, we will examine this proposal by studying a family of *interacting* topological phases, i.e., symmetry protected topological (SPT) phases[10–12]. The SPT phases possess bulk energy gaps and do not break any symmetry of the system, but have robust gapless edge excitations. These states can not be continuously connected to a trivial (vacuum) state without either breaking the protecting symmetry or closing the energy gap. If the protecting symmetry is preserved, both SPT phase and the trivial state have the same symmetry, but differ in the way the protecting symmetry is represented by their boundary excitations. So there exists a topological phase transition between the SPT phase and the trivial phase, and the corresponding critical theory does

not belong to the conventional Landau-Ginzburg-Wilson paradigm for symmetry breaking phase transitions. Such a critical point is dubbed “deconfined quantum critical point”. Thus, a crucial question is how to extract the critical properties from the ground state wave function of the SPT phase.

The simplest example of the SPT phases is the Haldane gapped phase of the antiferromagnetic spin-1 chain[13], which is protected by any one of the following discrete symmetries: time reversal symmetry, link inversion symmetry, or the $D_2 \simeq Z_2 \times Z_2$ symmetry comprising π rotations about two orthogonal axes[8, 14]. According to the classification theory, there exists only one nontrivial SPT phase. It is well-known that a fixed point wave function of the Haldane phase is given by the Affleck-Kennedy-Lieb-Tasaki (AKLT) valence bond solid state[15]. In this valence-bond solid picture, the important feature of the Haldane gapped state is the presence of fractionalized spin-1/2 edge excitations - the “confined spinons”.

For our purpose to generate a bulk critical spectrum that describes the continuous phase transition from the nontrivial Haldane phase to a trivial (vacuum) phase, the deconfined spinons should be emerged as coherent elementary excitations of the effective critical theory[16]. Hence, we introduce a symmetric extensive bipartition and require that the size of sublattice unit cell is larger than the correlation length of the AKLT state $\xi \approx 0.91024$. So the simplest choice is to divide the system into two sublattices both of which have two sites per unit cell. Then we expect that the critical theory can be characterized by a delocalized version of the edge spinons of the Haldane phase. This result is supported by our numerical finding. For a periodic AKLT chain, a critical and extensive ES with a residual entropy per site is obtained $s_r = 0.67602$, compared to $\ln 2$ for the AKLT

gapped state under usual left-right bipartition.

From the wave function corresponding to the lowest entanglement energy level, we can further calculate the nested entanglement entropy[17, 18], from which the central charge of the critical point can be precisely extracted $c \approx 1.01 \pm 0.01$. The corresponding critical theory can be identified as the same effective field theory as the spin-1/2 antiferromagnetic Heisenberg chain or the Haldane-Shastry spin model with inverse square long-range interaction[19], namely the (1+1) dimensional SU(2) level-1 Wess-Zumino-Witten field theory.

Model and method. The spin-1 AKLT parent Hamiltonian on a periodic chain is defined by[15]

$$H = J \sum_i \left[\mathbf{s}_i \mathbf{s}_{i+1} + \frac{1}{3} (\mathbf{s}_i \mathbf{s}_{i+1})^2 \right]. \quad (1)$$

The corresponding exact ground state wave function for $J > 0$ is expressed as

$$|\Psi\rangle = \sum_{\{s_i\}} \text{Tr} \left(\mathbf{A}^{[s_1]} \mathbf{A}^{[s_2]} \dots \mathbf{A}^{[s_L]} \right) |s_1, s_2, \dots, s_L\rangle, \quad (2)$$

where $s_i = -1, 0, +1$ are the local physical spin states and the local matrices are given by

$$\begin{aligned} A^{[-1]} &= \begin{pmatrix} 0 & 0 \\ -\sqrt{\frac{2}{3}} & 0 \end{pmatrix}, A^{[0]} = \begin{pmatrix} -\frac{1}{\sqrt{3}} & 0 \\ 0 & \frac{1}{\sqrt{3}} \end{pmatrix}, \\ A^{[+1]} &= \begin{pmatrix} 0 & \sqrt{\frac{2}{3}} \\ 0 & 0 \end{pmatrix}. \end{aligned} \quad (3)$$

In the thermodynamic limit, the spin-spin correlation function decays exponentially with a correlation length $\xi = 1/\ln 3 \simeq 0.91024$, and any two spins on the even number of the lattice sites are antiferromagnetically correlated. When we make a cut in real space, the periodic spin chain transforms into an open chain with fractionalized spin-1/2 edge spins. For a sufficient long length, these two edge spins are almost free, leading to four degenerate ground states. Actually each edge spin contributes two fold degeneracies. Thus, the degenerate ground state has a residual entropy per edge given by $s_r = \ln 2$.

For a general many-body wave function, the quantum entanglement properties between the subsystem A and subsystem B are characterized by the reduced density matrix ρ_A , which is formally written as the "thermal" density matrix of an entanglement Hamiltonian,

$$\rho_A = \text{Tr}_B |\Psi\rangle\langle\Psi| = e^{-H_E}, \quad (4)$$

where the wave function $|\Psi\rangle$ is assumed to be normalized. The full set of eigenvalues of H_E denoted by $\{\zeta_m\}$ constitutes the ES of subsystem A. These eigenvalues are directly related to the coefficients in the Schmidt decomposition of the ground state wave function. Among all entanglement states in subsystem A, those states with small eigenvalues have the larger weights in the ground

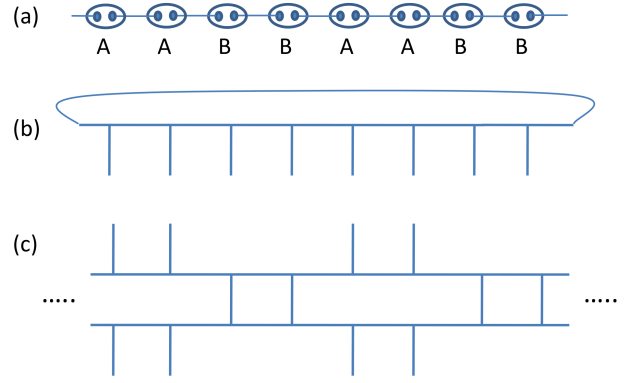


FIG. 1: (Color online) (a) The symmetric extensive bipartition chosen for the spin-1 AKLT model Hamiltonian. The small dots are denoted the spin-1/2 degrees of freedom and two spin-1/2 operators form the spin-1 objects. (b) The matrix product state as the exact ground state wave function. (c) The reduced density matrix upon tracing out the sublattice B.

state. The von Neumann entanglement entropy can be calculated by

$$S = -\text{Tr}(\rho_A \ln \rho_A) = \sum_m \zeta_m e^{-\zeta_m}. \quad (5)$$

For the AKLT wave function, when the open spin chain is further cut in the middle of the chain, and two symmetric subsystems A and B represent the left and right halves of the chain. Then we can easily calculate the ES, resulting a *single* entanglement energy level $\zeta = \ln 2$ with two fold degeneracies[20]. This implies that there exist degenerate edge excitations localized around the position where the cut is made. So such an ES from the left-right bipartition just produces single physical edge spectrum of the Haldane gapped phase.

In order to reveal the critical properties from the AKLT wave function, we somehow have to introduce extensive edge modes and make them delocalized inside the bulk system. So the simplest symmetric extensive bipartition is proposed and displaying in Fig.1a, where the subsystems A and B are related by a translation or reflection symmetry. Since the correlation length of the spin-1 AKLT state is just less than one lattice spacing, the lattice system is divided into two sublattices A and B, both of which contain two lattice sites per unit cell.

Consider a finite periodic spin chain with the length $L = 4N$. The exact ground state wave function has the matrix product form and is denoted in Fig.1b. According to the above symmetric extensive partition, the wave function basis are regrouped as

$$\begin{aligned} &|s_1, s_2, \dots, s_L\rangle \\ &= |s_1 s_2, s_5 s_6, \dots, s_{4N-3} s_{4N-2}\rangle |s_3 s_4, s_7 s_8, \dots, s_{4N-1} s_{4N}\rangle. \end{aligned}$$

Then we can formally regard $(s_1 s_2, s_5 s_6, \dots, s_{4N-3} s_{4N-2})$ and $(s_3 s_4, s_7 s_8, \dots, s_{4N-1} s_{4N})$ as two groups of indices for

a large matrix $M = \text{Tr}(\mathbf{A}^{[s_1]}\mathbf{A}^{[s_2]}\dots\mathbf{A}^{[s_{4N}]})$, and the wave function is simplified as

$$|\Psi\rangle \equiv \sum_{i,j} M_{i,j} |\psi_A^i\rangle |\psi_B^j\rangle, \quad (6)$$

where $|\psi_A^i\rangle$ and $|\psi_B^j\rangle$ represent the orthonormal basis of the subsystems A and B, respectively. So the ground state density matrix is expressed as

$$\rho = \sum_{i,j} \sum_{m,n} M_{i,j} M_{m,n}^* |\psi_A^i\rangle |\psi_B^j\rangle \langle \psi_A^m| \langle \psi_B^n|. \quad (7)$$

When all the degrees of freedom of the subsystem B are traced out, the reduced density matrix is given by

$$\rho_A = \sum_{i,m} (MM^\dagger)_{i,m} |\psi_A^i\rangle \langle \psi_A^m|, \quad (8)$$

which is represented by Fig.1c. To get the eigenvalues of ρ_A , it is more convenient to do singular value decomposition: $M = U\Lambda V^\dagger$, where Λ is a real diagonal matrix. Since $MM^\dagger = U\Lambda V^\dagger V\Lambda U^\dagger = U\Lambda^2 U^\dagger$, Λ^2 gives rise to the eigenvalues of ρ_A and hence the entanglement energy levels $\{\zeta_m\}$ are obtained.

Numerical results.- The entanglement energy levels ζ_m are calculated numerically for different length of the chain $L = 4, 8, 12, 16, 20, 24, 28$. Here we have kept *all* the entanglement energy states. The corresponding entanglement spectra are displayed in Fig.2. For each system size, the lowest entanglement energy level ζ_0 is always singlet, corresponding to a nondegenerate ground state, while the second entanglement level ζ_1 is always triplet, representing the first three-fold excited state. The energy differences between any two entanglement energy levels decrease as the system size grows. There is no apparent energy gap in the ES, as indicated in the largest system size $L = 28$. Importantly, we notice that the total number of entanglement energy levels increases as $2^{L/2}$, suggesting that the fundamental objects in the reduced subsystem A with a length $L_r = L/2$ are characterized by the spin-1/2 degrees of freedom living on the bonds instead of the lattice sites.

The lowest entanglement energy level with the largest weight ξ_0 represents the ground state energy of the entanglement Hamiltonian H_E . Although the original AKLT model Hamiltonian only includes the nearest neighbor interactions, the entanglement Hamiltonian H_E generally involves the long-range interactions. In Fig.3a, the numerical result of ξ_0 is shown as a function of system size L . The numerical data can be fitted by $\xi_0 = 0.27L - 0.17$. The constant term can become even smaller when data of small size systems is neglected. In the thermodynamic limit, ξ_0 is expected to depend linearly on the system size, suggesting that the subsystem A is extensive. Typically, the difference of the lowest two entanglement energy levels $\xi_1 - \xi_0$ is displayed as a function of the inverse lattice size in Fig.3b, and it can be fitted by a power law $L^{-\alpha}$ with $\alpha = 0.957 \pm 0.003$. Similarly, the differences of any

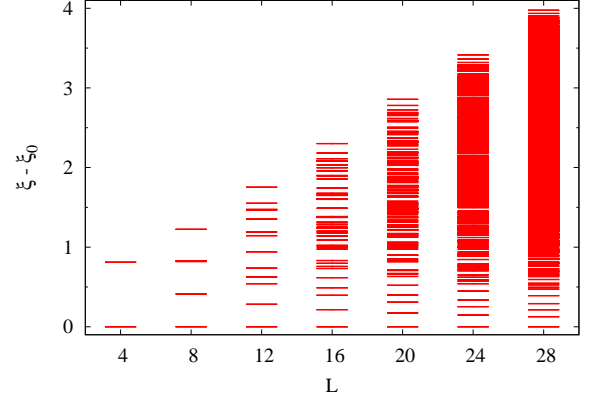


FIG. 2: (Color online) Entanglement spectrum of the spin-1 AKLT ground state wave function under a symmetric extended partition for different length of the chain.

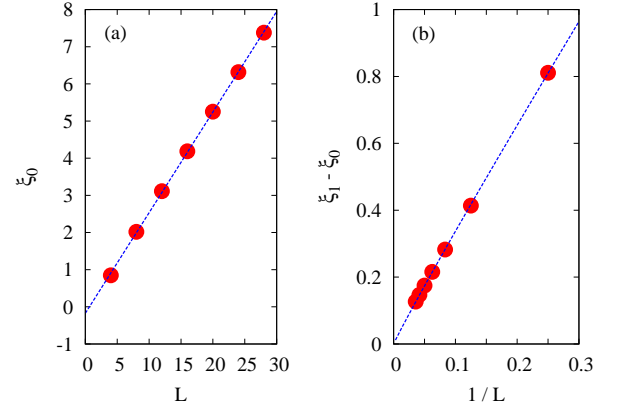


FIG. 3: (Color online) (a) The lowest entanglement energy level ξ_0 as a function of system size L . (b) The difference of the lowest two entanglement energies $\xi_1 - \xi_0$ as a function of the inverse system size.

other two levels vanish in the same limit. Such a behavior suggests that the resulting ES becomes gapless in the thermodynamic limit. Thus, the ES from the symmetric extensive bipartition for the AKLT ground state wave function is extensive and gapless, providing the critical properties of the topological phase.

To describe this bulk critical ES, we calculate the von Neumann entanglement entropy per lattice site $s_r = S/L_r$. The numerical result is shown in Fig.4. As the system size increases, the entropy quickly saturates at $s_r = 0.676$, within 5% from the free spin-1/2 entropy $\ln 2$. This seems to suggest that the partitioned subsystem A consists of extensive delocalized spin-1/2 degrees of freedom, namely, the edge spinons of the Haldane gapped phase is delocalized throughout the bulk system.

Universality class of the critical point.- In order to determine the universality class of the critical point from the ES, it is noticed that the reduced density matrix ρ_A and the entanglement Hamiltonian H_E share the same

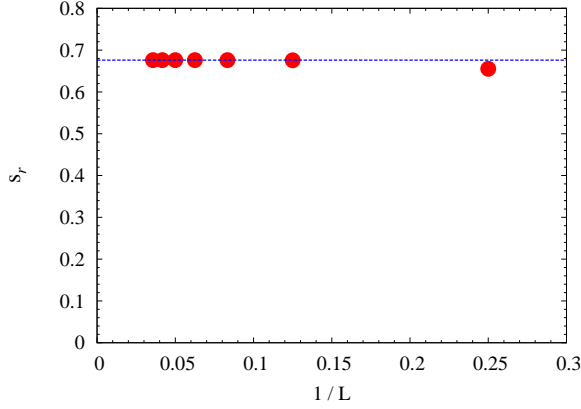


FIG. 4: (Color online) The von Neumann entanglement entropy per lattice site. As system size increases, the entropy saturates at 0.676.

”ground state” wave function:

$$\rho_A|\Psi_0\rangle = e^{-\zeta_0}|\Psi_0\rangle, \quad H_E|\Psi_0\rangle = \zeta_0|\Psi_0\rangle. \quad (9)$$

Actually, we can further divide the system described by H_E with the length L_r into two subsystems, where the length of one subsystem is l and the other is $L_r - l$. A so-called nested entanglement matrix can be introduced as[17, 18]

$$\rho(l) = \text{Tr}_{l+1, l+2, \dots, L_r} (|\Psi_0\rangle\langle\Psi_0|), \quad (10)$$

where $\text{Tr}_{l+1, l+2, \dots, L_r}$ means the tracing out over the degrees of freedom of the subsystem with the length $L_r - l$. The nested entanglement entropy can be defined by

$$s(l, L_r) = -\text{Tr}_{1, 2, \dots, l} [\rho(l) \ln \rho(l)], \quad (11)$$

where the trace is taken over the states in the subsystem with the length l . At the critical point, according to the conformal field theory[21], the nested entanglement entropy for the systems with periodic boundary condition is described by

$$s(l, L_r) = \frac{c}{3} \ln \left[\frac{L_r}{\pi} \sin \left(\frac{\pi l}{L_r} \right) \right] + s_0, \quad (12)$$

where c is the central charge of the entanglement Hamiltonian and s_0 is a non-universal constant. The central charge underpins the conformal field theory of the critical point.

The numerical result for the nested entanglement entropy $s(l, L_r)$, resulting from bipartition of the ground state wave function $|\Psi_0\rangle$ is displayed as a function of $g(l, L_r) = \frac{L_r}{\pi} \sin \left(\frac{\pi l}{L_r} \right)$ in Fig.5. The data can be well fitted with the central charge $c = 1.01 \pm 0.01$. In fact, from the critical ES, we can further uncover the subset operator content of the critical point beyond the central charge[22]. Here we would like to emphasize that the

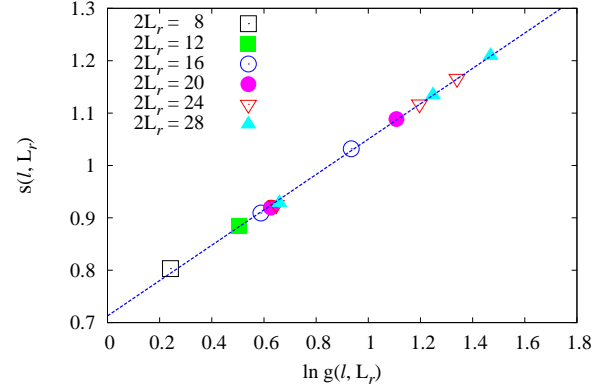


FIG. 5: (Color online) The nested entanglement entropy $s(l, L_r)$, as a function of $g(l, L_r)$, where L_r is the size of the subsystem, while l and $L_r - l$ are the lengths of the two nested subsystems.

central charge obtained by the nested entanglement entropy can determine the critical properties of the entanglement Hamiltonian. The entanglement Hamiltonian may thus represent the model Hamiltonian of the critical point of the phase transition between the nontrivial Haldane phase and the trivial (vacuum) phase. Moreover, we can further find that the lower-lying entanglement energy part resembles the excitation spectrum of the low-energy theory for the spin-1/2 antiferromagnetic Heisenberg chain, more precisely, that for the antiferromagnetic spin-1/2 Haldane-Shastry model with inverse square long-range interaction[19]. According to the conformal field theory, the effective theory of both models is characterized by the (1+1) dimensional $SU(2)$ level-1 Wess-Zumino-Witten conformal field theory[23], where the elementary excitations are the ”spinon” particles, satisfying the semion statistics[24]. Actually such a gapless critical theory can also be regarded as the edge theory of a (2+1) dimensional SPT phase[25].

Discussion and Conclusion.— Through the analysis of quantum entanglement properties with a symmetric extensive bipartition of the AKLT wave function, the information on the critical point of the topological Haldane phase to the trivial gapped phase can intriguingly emerge, as long as the size of the sublattice unit cell is larger than the correlation length of the Haldane phase. Other types of symmetric extensive bipartitions have also been considered, leading to the similar results. This further implies that it is the structure of the edge excitations of the bulk topological phase determines the minimal size of the sublattice unit cell. Like other entanglement-based methods, the selection of states according to the eigenvalues of the reduced density matrix effectively distills the topological (edge) degree of freedom as the leading contributions, which are insensitive to the size of the sublattice unit cell. This guarantees the general applicability of our method to models with larger correlation length, as long as the edge spectrum of the bulk phase can be

described by a small dimensional space, such as in a SPT phase.

We also found that any *asymmetric* extensive bipartition for the AKLT wave function always leads to an ES with multiple gaps. So the gapped excitation spectrum of the Haldane gapped phase can not be obtained through the asymmetric extensive bipartition. Moreover, we have noticed that the symmetries of the model Hamiltonian for the SPT phase are relevant in determining the corresponding effective field theory of the critical point. Finally an important correspondence between the critical theory and its gapless boundary theory of the Haldane gapped phase can be established generally. The boundary theory of the Haldane gapped phase is the critical theory spatially confined between the topological phase and the trivial gapped phase[26], while the coherence between extensive edge excitations (deconfined spinons) emerges as the bulk critical theory of the topological Haldane phase.

For a general SPT phase in two dimension, when an appropriate symmetric extensive (checkerboard) biparti-

tion for the ground state wave function is introduced and satisfy the condition that the characteristic length of the sublattice unit cell is larger than the correlation length of the topological SPT phase, a critical ES in the same spatial dimension can be obtained, and the effective critical field theory of its phase transition to the trivial gapped phase can be determined. The resulting critical theory is of topological in nature. Beyond the SPT phases, like Hsieh and Fu[9] used a single-particle Chern insulator to derive a bulk critical ES describing the phase transition of the integer quantum Hall affect, a conceptually similar but richer scenario may also be expected in fractional quantum Hall systems[27, 28]. These related works are under investigations.

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